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LONGWAVE APPROXIMATION MODEL FOR GAS SHEAR FLOW IN A CHANNEL OF VARYING AREA

V. M. Teshukov

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An approximate system of equations that describe unsteady flow of an inviscid non-heat-conducting gas in a narrow channel of varying area is derived. Generalized characteristics and hyperbolicity conditions are obtained for this system of equations. In connection with characteristics theory, the average Mach number and the flow criticality condition are introduced. Exact solutions that describe steady transonic channel flows are investigated.

Plane-parallel and axisymmetric unsteady flows in a long channel of varying area are studied. It is assumed that the channel entrance flow parameters are nonuniformly distributed over the cross section. A mathematical model of longwave approximation that extends the well-known channel approximation equations [1, 2] to the case of inhomogeneous flows is developed. The characteristics of the integrodifferential equations governing the flow evolution are studied using the method developed in [3, 4], and hyperbolicity conditions are obtained. A class of stationary solutions that describe inhomogeneous transonic flows in a channel of varying area is constructed. Exact solutions that model flow separation from the wall and formation of a return-flow zone are obtained.

Steady varying-area flows of a polytropic gas and isentropic flows have been investigated [5, 6] within the framework of a similar approximation. In the present paper, steady flows with a critical layer are analyzed using some methods of analysis developed in [5].

1. Longwave Approximation Model. We study plane-parallel flow of an inviscid gas in a long channel of varying area. Let X and Y be Cartesian plane coordinates, T time, and the equations $Y = 0$ and $Y = H_0A(XL_0^{-1})$ define the lower and upper walls of the channel (H_0 is the characteristic width of the channel and L_0 is the characteristic length). In what follows, we assume that $\varepsilon = H_0L_0^{-1} \ll 1$.

We introduce the following dimensionless independent and dependent variables:

$$x = L_0^{-1}X, \quad y = H_0^{-1}Y, \quad t = L_0^{-1}U_0T,$$
$$u = U_0^{-1}U, \quad v = H_0^{-1}L_0U_0^{-1}V, \quad \rho = R_0^{-1}R, \quad p = R_0^{-1}U_0^{-2}P. \tag{1.1a}$$

Here R_0 and U_0 are the characteristic density and horizontal velocity, U, V, R, and P are the dimensional velocity components, density, and pressure, the small letters denote the corresponding dimensionless quantities. The gas-dynamic equations are represented in the dimensionless variables as

$$u_t + uu_x + vu_y + \rho^{-1}p_x = 0, \quad \rho_t + (u\rho)_x + (v\rho)_y = 0,$$
$$\varepsilon^2(v_t + uv_x + vv_y) + \rho^{-1}p_y = 0, \quad S_t + uS_x + vS_y = 0, \quad \rho = \rho(p, S). \tag{1.1}$$

The last equality defines the equation of state (S is the entropy).

Longwave approximation equations are obtained when $\varepsilon \to 0$ in (1.1). In the limit, the law of conservation of vertical momentum gives the equality $p_y = 0$, and, hence, $p = p(x,t)$. This equality expresses the fact that the pressure distribution across the channel levels off much more rapidly than the

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 1, pp. 15–27, January-February, 1998. Original article submitted May 30, 1996.

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same distribution along the channel (because of the considerable difference in geometric scales). In the approximate description, the transverse pressure distribution levels off instantaneously. In this case, the transverse distributions of the gas entropy and density may not be uniform. Clearly, this approximation applies for smooth distributions, and modeling of flows with sharp gradients and jumps of parameters requires separate consideration. Integration of the continuity equation yields

$$v = -(\rho(p,S))^{-1} \int_0^y ((\rho(p,S))_t + (u\rho(p,S))_x)\,dy, \tag{1.2}$$

where the boundary condition $v = 0$ for $y = 0$ is used.

The boundary condition on the upper wall of the channel $v - A'(x)u = 0$ with allowance for (1.2) is written as

$$\Big(\int_0^{A(x)} \rho(p,S)\,dy\Big)_t + \Big(\int_0^{A(x)} u\rho(p,S)\,dy\Big)_x = 0. \tag{1.3}$$

Equations (1.2) and (1.3) together with the equations

$$u_t + uu_x + vu_y + (\rho(p,S))^{-1}p_x = 0, \quad S_t + uS_x + vS_y = 0 \tag{1.4}$$

form a closed system of equations for determining $u(x,y,t)$, $S(x,y,t)$, $p(x,t)$, and $v(x,y,t)$.

For further consideration, system (1.1) is conveniently transformed to the Eulerian–Lagrangian coordinates x', t', and λ ($\lambda \in [0,1]$) using the relations

$$x = x', \quad t = t', \quad y = \Phi(x',\lambda,t'), \quad \Phi_t + u(x,\Phi,t)\Phi_x = v(x,\Phi,t), \quad \Phi(x,\lambda,0) = \Phi_0(x,\lambda).$$

Solution of the above Cauchy problem gives the evolution of the contact characteristics of system (1.1), each of which is defined by the equation $y = \Phi(x,\lambda,t)$ with fixed Lagrangian variable λ. The relation $y = \Phi_0(x,\lambda)$ gives the initial position of the contact characteristic $\lambda = \text{const}$. The function $\Phi_0(x,\lambda)$ is chosen so that $\Phi_0(x,0) = 0$ and $\Phi_0(x,1) = A(x)$ [in particular, one can set $\Phi_0(x,\lambda) = \lambda A(x)$]. In the new variables (below, the prime is dropped), the gas-dynamic equations take the form

$$\begin{gathered} \rho(u_t + uu_x) + p_x - \Phi_x(\Phi_\lambda^{-1})p_\lambda = 0, \quad H_t + (uH)_x = 0, \\ \varepsilon^2\rho(v_t + uv_x) + \Phi_\lambda^{-1}p_\lambda = 0, \quad S_t + uS_x = 0. \end{gathered} \tag{1.5}$$

Here the sought function $H = \rho\Phi_\lambda$ is introduced. The boundary conditions on the channel walls are equivalent to the relations

$$\Phi(x,0,t) = 0, \quad \int_0^1 (\rho(p,S))^{-1}H\,d\lambda = \int_0^1 \Phi_\lambda\,d\lambda = A(x). \tag{1.6}$$

System (1.5) and (1.6) supplemented by the equation

$$\Phi_t + u\Phi_x = v \tag{1.7}$$

forms the exact model of channel gas flow. In the longwave approximation model, $p_\lambda = 0$, i.e., $p = p(x,t)$. Equations (1.5) for $\varepsilon = 0$ reduce to the following integrodifferential system:

$$u_t + uu_x + (\rho\sigma)^{-1}\Big[\int_0^1 \frac{H_x}{\rho}\,d\lambda - \int_0^1 \frac{H\rho_S}{\rho^2}S_x\,d\lambda - A'(x)\Big] = 0, \quad H_t + (uH)_x = 0, \quad S_t + uS_x = 0 \tag{1.8}$$

$$\Big(\rho_p = \frac{\partial}{\partial p}\rho(p,S) = c^{-2},\ \rho_S = \frac{\partial}{\partial S}\rho(p,S),\ \sigma = \int_0^1 \frac{H\rho_p}{\rho^2}\,d\lambda = \int_0^1 \frac{H}{\rho^2c^2}\,d\lambda\Big).$$

Here c is the speed of sound.

Equations (1.8) form a closed system for determining the functions $u(x,\lambda,t)$, $H(x,\lambda,t)$, and $S(x,\lambda,t)$. In this case, the nonlocal dependence of p on H and S is given by (1.6). If a solution of system (1.8) is obtained, the vertical component of the velocity vector is determined from (1.7) and $\Phi(x,\lambda,t)$ is given by the equation

$$\Phi = \int_0^\lambda (\rho(p,S))^{-1} H\,d\lambda.$$

For a polytropic gas $[\rho = p^{1/\gamma} b(S)]$, Eq. (1.6) is solved explicitly for p:

$$p = (A(x))^{-\gamma}\Big(\int_0^1 (b(S))^{-1} H\,d\lambda\Big)^\gamma.$$

Note that for the class of particular solutions characterized by the equalities $u_\lambda = 0$, $S_\lambda = 0$, and $H_\lambda = 0$, system (1.8) reduces to the well-known channel-approximation equations:

$$\rho(u_t + uu_x) + p_x = 0, \quad (\rho A)_t + (u\rho A)_x = 0, \quad S_t + uS_x = 0. \tag{1.9}$$

In this case, $y = \Phi = \lambda A(x)$. Hence, system (1.8) [or (1.2)–(1.4)] is a generalization of the well-known channel approximation model to the case of inhomogeneous flows of varying area.

The nonzero component of the curl of the velocity in the chosen scales is represented as $\omega = V_X - U_Y = U_0 H_0^{-1}(\varepsilon^2 v_x - u_y)$.

In the approximate theory, $\omega = -U_0 H_0^{-1} u_y$, and, hence, flows whose velocities u do not depend on λ are irrotational $[u_y = u_\lambda(\Phi_\lambda)^{-1} = 0]$, and the general flows governed by Eqs. (1.8) are rotational.

2. Hyperbolicity of the Basic System of Equations. System (1.8) describes processes with a finite velocity of perturbation propagation along the channel axis. This velocity is found using special integral relations that take into account flow inhomogeneity. The exact description of perturbation propagation is based on the definitions of the characteristics of the system of equations with operator coefficients formulated in [3].

We represent Eqs. (1.8) in the form

$$\mathbf{U}_t + B\langle \mathbf{U}_x\rangle = \mathbf{f}, \tag{2.1}$$

where $\mathbf{U}(x,\lambda,t) = (u,H,S)^t$ $[(\ldots)^t$ denotes transposition]. The action of the operator B on any smooth function $\varphi = (\varphi_1,\varphi_2,\varphi_3)^t$ is given by the formula

$$B\langle\varphi\rangle = \begin{pmatrix} u\varphi_1 + (\rho\sigma)^{-1}\Big(\int_0^1 \frac{\varphi_2}{\rho}\,d\nu - \int_0^1 \frac{H\rho_S}{\rho^2}\varphi_3\,d\nu\Big) \\ H\varphi_1 + u\varphi_2 \\ u\varphi_3 \end{pmatrix}. \tag{2.2}$$

In accordance with [3, 4], the differential equation $dx/dt = k(x,t)$ defines the characteristic curve $x = x(t)$ of system (1.8) if the eigenvalue problem

$$(\mathbf{F}, B\langle\varphi\rangle) = k(\mathbf{F},\varphi) \tag{2.3}$$

has a nontrivial solution. Here k is an eigenvalue, $\mathbf{F} = (F_1,F_2,F_3)$ is the sought eigenvector functional, $(\mathbf{F},\varphi)$ is the action of the functional $\mathbf{F}$ on the smooth trial function φ. Here $\mathbf{F}$ is treated as a liner functional that acts on functions of the variable λ. In the case considered, the quantities entering (2.3) bear a parametric relationship to x and t. The equality

$$(\mathbf{F}, \mathbf{U}_t + B\langle\mathbf{U}_x\rangle) = (\mathbf{F}, \mathbf{U}_t + k\mathbf{U}_x) = (\mathbf{F},\mathbf{f}) \tag{2.4}$$

is called the characteristic relation [in (2.4) the vector $\mathbf{U}$ is differentiated only in the direction $dx/dt = k$ in the x,t plane]. System (2.1) is a hyperbolic system of equations if all k^α satisfying (2.3) are real and the set of corresponding eigenfunctionals $\{\mathbf{F}^\alpha\}$ is complete in the following sense: if the function φ is sufficiently smooth, it follows from $(\mathbf{F}^\alpha, \varphi) = 0$ that $\varphi = 0$. Then the characteristic relations (2.4) are equivalent to the initial system (2.1).

We obtain the hyperbolicity condition for system (1.8) in the case where u is a monotonic function of λ ($u_\lambda \neq 0$). Since the trial functions φ_1, φ_2, and φ_3 are independent, Eq. (2.3) is equivalent to the following equalities:

$$(F_1, (u-k)\varphi_1) + (F_2, H\varphi_1) = 0, \quad (F_2, (u-k)\varphi_2) + \sigma^{-1}\int_0^1 \varphi_2\rho^{-1}\,d\nu (F_1, \rho^{-1}) = 0,$$
$$(F_3, (u-k)\varphi_3) - \sigma^{-1}\int_0^1 \rho_S \rho^{-2} H\varphi_3\,d\nu (F_1, \rho^{-1}) = 0. \tag{2.5}$$

In accordance with (2.5), the action of the functional F_2 on any trial function ψ is uniquely determined if the functional F_1 is known:

$$(F_2, \psi) = -(F_1, (u-k)H^{-1}\psi). \tag{2.6}$$

Here ψ is a trial function and $H \neq 0$. The functional F_1, as follows from (2.5) and (2.6), must satisfy the equation

$$(F_1, (u-k)^2 H^{-1}\varphi_2) - \sigma^{-1}\int_0^1 \varphi_2\rho^{-1}\,d\nu (F_1, \rho^{-1}) = 0. \tag{2.7}$$

For $k \neq u(x, \lambda, t)$, where $\lambda \in [0, 1]$, the action of the functional F_1 on the trial function ψ can be defined by

$$(F_1, \psi) = \sigma^{-1}\int_0^1 H\rho^{-1}(u-k)^{-2}\psi\,d\nu (F_1, \rho^{-1}). \tag{2.8}$$

Substituting $\psi = \rho^{-1}$, we obtain the existence condition for a nontrivial solution of Eq. (2.8), i.e., the following equation for determining the characteristic velocity k:

$$\chi(k) = \int_0^1 \frac{H}{\rho^2 c^2}\,d\nu - \int_0^1 \frac{H}{\rho^2 (u-k)^2}\,d\nu = 0. \tag{2.9}$$

As a result, for each root k_i of Eq. (2.9) we obtain the eigenfunctional F_1^i, whose action on an arbitrary function ψ is defined by formula (2.8) for $(F_1^i, \rho^{-1}) = 1$ [this equality follows from (2.9)]. Analysis of the characteristic equation (2.9) shows that there are two characteristic roots on the real axis: $k = k_1 > \max\limits_\lambda u(x, \lambda, t) = u_1$ and $k = k_2 < \min\limits_\lambda u(x, \lambda, t) = u_0$. Indeed, $\chi'(k) \neq 0$ for $k > u_1$ and $k < u_0$, $\chi(\pm\infty) > 0$, and $\chi(k) \to -\infty$ for $k \to u_0$ and $k \to u_1$ (for smooth dependences of H, ρ, and u on λ). If the functions H, ρ, u, and c do not depend on λ, Eq. (2.9) becomes the ordinary equation of sonic characteristics for system (1.9). Therefore, the characteristics that correspond to the roots k_1 and k_2 are analogs of sonic characteristics. In the general case, Eq. (2.9) can have complex roots. To formulate conditions that guarantee the absence of complex roots for the given solution u, H, S, we examine the analytic continuation of the function χ to the complex plane. Similar reasoning to that in [4] provides the following result. Equation (2.9) has only real roots ($k = k_1$ and $k = k_2$) if

$$\chi^+(u) \neq 0, \quad \Delta \arg \chi^+(u)/\chi^-(u) = 0. \tag{2.10}$$

Here $\chi^\pm(u)$ are the limiting values of the analytic function $\chi(z)$ on the segment $[u_0, u_1]$ from the upper and lower half-planes; $\Delta\arg f$ is the increment of the complex function f on the indicated segment. When the first component F_1^i of the eigenvector functional that corresponds to the characteristic root k_i is determined, we define the second component F_2^i by formula (2.6) and F_3^i by the formula

$$(F_3^i, \psi) = \sigma^{-1} \int_0^1 \rho_S \rho^{-2} \frac{H\psi}{u - k_i}\, d\nu$$

[see (2.5)].

Analysis shows that nontrivial solutions of Eq. (2.7) in the class of generalized functions can be found for an infinite set of eigenvalues $k^\lambda = u(x, \lambda, t)$, where $\lambda \in (0, 1)$ (a continuous spectrum of characteristic velocities). Let $\delta(\nu - \lambda)$ be the generalized Dirac delta function and $\delta'(\nu - \lambda)$ its derivative. The action of these functionals on an arbitrary function φ is defined by the standard formulas

$$(\delta(\nu - \lambda), \varphi(\nu)) = \varphi(\lambda), \quad (\delta'(\nu - \lambda), \varphi(\nu)) = -\varphi'(\lambda).$$

Equation (2.7) holds if we set $F_1^{\lambda 1} = \rho(\nu)\delta'(\nu - \lambda)$. Indeed,

$$(F_1^{\lambda 1}, (\rho(\nu))^{-1}) = (\delta'(\nu - \lambda), 1) = 0,$$

$$(F_1^{\lambda 1}, (u(\nu) - u(\lambda))^2 (H(\nu))^{-1} \varphi_2(\nu)) = (\delta'(\nu - \lambda), (u(\nu) - u(\lambda))^2 \rho\varphi_2(\nu)(H(\nu))^{-1}) = 0.$$

The second component of the vector functional $F_2^{\lambda 1} = u_\lambda(\lambda)\rho(\lambda)(H(\nu))^{-1}\delta(\nu - \lambda)$ is calculated by substituting the first component $F_1^{\lambda 1}$ into formula (2.6), and the third component $F_3^{\lambda 1}$ can be taken equal to zero. To construct one more solution of Eq. (2.7), we introduce the functional P^λ, which acts on an arbitrary function $\varphi(\nu)$ as follows:

$$(P^\lambda, \varphi(\nu)) = \int_0^1 \frac{H(\nu)}{\rho^2(\nu)} \frac{\rho(\nu)\varphi(\nu) - \rho(\lambda)\varphi(\lambda)}{(u(\nu) - u(\lambda))^2}\, d\nu.$$

Here by the integral we mean the principal value. The functional P^λ satisfies the relation

$$\left(P^\lambda, \frac{(u(\nu) - u(\lambda))^2}{H(\nu)} \varphi(\nu)\right) = \int_0^1 \frac{\varphi(\nu)}{\rho(\nu)}\, d\nu.$$

A solution of Eq. (2.7) for $k = k^\lambda(x, t) = u(x, \lambda, t)$ is sought in the form $F_1 = F_1^{\lambda 2} = C(\lambda)\delta(\nu - \lambda) + P^\lambda$. It turns out that it suffices to set $C(\lambda) = \sigma\rho(\lambda)$ to satisfy this equation.

As previously, the action of the second component of the eigenvector functional $F_2^{\lambda 2}$ is defined by formula (2.6):

$$(F_2^{\lambda 2}, \varphi(\nu)) = -\int_0^1 \frac{\varphi(\nu)\, d\nu}{\rho(\nu)(u(\nu) - u(\lambda))}.$$

It is easy to verify that if the third component of the eigenvector functional $F_3^{\lambda 2}$ is defined by the relation

$$(F_3^{\lambda 2}, \varphi(\nu)) = -\int_0^1 \frac{\rho_S(\nu)}{\rho^2(\nu)} \frac{H(\nu)\varphi(\nu)}{u(\nu) - u(\lambda)}\, d\nu,$$

the last equation of (2.5) also holds.

Besides the vector functionals $\mathbf{F}^{\lambda 1}$ and $\mathbf{F}^{\lambda 2}$ constructed above, $\mathbf{F}^{\lambda 3} = (0, 0, \delta(\nu - \lambda))$ also satisfies Eqs. (2.5) for $k^\lambda = u(x, \lambda, t)$. Here and below, for brevity, the arguments x and t in the notation for the sought functions are dropped.

Acting on Eqs. (1.8) by the vector functionals, we obtain the following characteristic relations:

$$\int_0^1 \frac{H(u_t + k_i u_x)}{\rho(u - k_i)^2}\, d\nu - \int_0^1 \frac{H_t + k_i H_x}{\rho(u - k_i)}\, d\nu + \int_0^1 \frac{\rho_S}{\rho^2} \frac{H(S_t + k_i S_x)}{u - k_i}\, d\nu = A'(x),$$

$$\rho_\lambda H(u_t + uu_x) + \rho H(u_{\lambda t} + uu_{\lambda x}) - \rho u_\lambda(H_t + uH_x) = 0, \tag{2.11}$$

$$\rho(\lambda)\sigma(u_t(\lambda)+u(\lambda)u_x(\lambda))+\int_0^1 \frac{H(\nu)}{\rho^2(\nu)}\,\frac{\rho(\nu)(u_t(\nu)+u(\lambda)u_x(\nu))-\rho(\lambda)(u_t(\lambda)+u(\lambda)u_x(\lambda))}{(u(\nu)-u(\lambda))^2}\,d\nu$$

$$-\int_0^1 \frac{H_t(\nu)+u(\lambda)H_x(\nu)}{\rho(\nu)(u(\nu)-u(\lambda))}\,d\nu+\int_0^1 \frac{H(\nu)\rho_S(\nu)}{\rho^2(\nu)}\,\frac{S_t(\nu)+u(\lambda)S_x(\nu)}{u(\nu)-u(\lambda)}\,d\nu = A'(x),$$

$$S_t(\lambda)+u(\lambda)S_x(\lambda)=0.$$

To complete the verification of the hyperbolicity conditions for system (1.8), we check the completeness of the resulting system of eigenfunctionals. Let $(\mathbf{F}^i,\varphi)=0$ ($i=1$ and 2) and $(\mathbf{F}^{\lambda j},\varphi)=0$ ($j=1$, 2, and 3). We show that $\varphi=0$ if u, H, and S satisfy (2.10). It follows from the equation $(\mathbf{F}^{\lambda 3},\varphi)=0$ that $\varphi_3=0$, and the equation $(\mathbf{F}^{\lambda 1},\varphi)=0$ gives the relation

$$H\,\frac{\partial}{\partial\lambda}(\rho\varphi_1)-\rho u_\lambda\varphi_2=0. \tag{2.12}$$

With allowance for (2.12), the equation $(\mathbf{F}^{\lambda 2},\varphi)=0$ is written as

$$\sigma\rho\varphi_1-\int_0^1 \frac{H(\nu)}{\rho^2(\nu)u_\nu(\nu)}\,\frac{\partial}{\partial\nu}\,\frac{\rho(\nu)\varphi_1(\nu)-\rho(\lambda)\varphi_1(\lambda)}{u(\nu)-u(\lambda)}\,d\nu=0. \tag{2.13}$$

Direct substitution of the function $\rho\varphi_{0i}=(u-k_i)^{-1}$ into (2.13) shows that it is a solution of this equation. Instead of the sought function φ_1 we introduce the new unknown

$$w=\rho\varphi_1-\sum_{i=1}^{2}\alpha_i\rho\varphi_{0i}, \tag{2.14}$$

choosing the coefficients $\alpha_i(x,t)$ from the conditions $w=0$ for $\lambda=0$ and $\lambda=1$. Integrating (2.13) by parts and passing to integration with respect to u, we obtain the following singular integral equation for determining w:

$$w\Big(\sigma+\frac{H(1)}{\rho^2(1)u_\lambda(1)(u(1)-u(\lambda))}-\frac{H(0)}{\rho^2(0)u_\lambda(0)(u(0)-u(\lambda))}$$

$$-\int_{u_0}^{u_1}\frac{\partial}{\partial\nu}\Big(\frac{H(\nu)}{\rho^2(\nu)u_\nu(\nu)}\Big)\frac{du}{u_\nu(\nu)(u-u(\lambda))}\Big)+\int_{u_0}^{u_1}\frac{\partial}{\partial\nu}\Big(\frac{H(\nu)}{\rho^2(\nu)u_\nu(\nu)}\Big)\frac{w(u)\,du}{u_\nu(\nu)(u-u(\lambda))}\Big)=0. \tag{2.15}$$

Let the coefficients of the singular integral equation (2.15) satisfy the Hölder condition for the variable u. According to the general theory of singular integral equations [7], Eq. (2.15) is uniquely solvable in the class of functions that satisfy the Hölder condition in the interval (u_0,u_1) and are bounded at the ends of the interval if the symbol of the equation does not degenerate and the index of the equation is equal to zero. These conditions reduce to (2.10), and, therefore, from (2.15) it follows that $w=0$. Hence,

$$\rho\varphi_1=\sum_{i=1}^{2}\alpha_i\rho\varphi_{0i}. \tag{2.16}$$

With allowance for (2.12), the equations $(\mathbf{F}^i,\varphi)=0$ are written as

$$\int_0^1 \frac{H}{u_\nu\rho^2}\,\frac{\partial}{\partial\nu}\Big(\frac{\rho\varphi_1}{u-k_1}\Big)\,d\nu=0,\qquad i=1,2. \tag{2.17}$$

By virtue of (2.9),

$$\int_0^1 \frac{H}{u_\nu\rho^2}\,\frac{\partial}{\partial\nu}\Big(\frac{1}{(u-k^i)(u-k^j)}\Big)\,d\nu=0 \tag{2.18}$$

for $i \neq j$. Substituting (2.16) into (2.17) and taking into account relations (2.18) yield equations from which it follows that $\alpha_i = 0$. Hence, $\varphi_1 = 0$, $\varphi_2 = 0$, and $\varphi_3 = 0$.

As a result, it is established that system (1.8) is hyperbolic if the solution (u, H, S) satisfies conditions (2.10) and $u_\lambda \neq 0$.

For nonmonotonic velocity profiles, one can obtain the hyperbolicity conditions for Eqs. (1.8) by changing somewhat the reasoning in [8], where a similar formulation of the problem was studied. Apparently, violation of the hyperbolicity conditions leads to incorrectness of the Cauchy problem for Eqs. (1.8). This fact can be established for the linearized system of equations with "frozen" coefficients. Indeed, if we consider linearization of Eqs. (1.8) on the vector $\mathbf{U}_0 = (u(x_0, \lambda, t_0),\ H(x_0, \lambda, t_0),\ S(x_0, \lambda, t_0))$, where $\mathbf{U} = (u, H, S)$ is a solution of system (1.8) (a linearized system with "frozen" coefficients of x and t), the homogeneous equations admit solutions of the form $\mathbf{U} = \mathbf{V}_0(\lambda) l^{-n}\, e^{il(x-kt)}$, where l is a real parameter, $n > 0$, and k is a complex root ($\operatorname{Im} k > 0$) of Eq. (2.9) for $\mathbf{U} = \mathbf{U}_0$. In the limit $l \to \infty$, we have $\mathbf{U}(x, \lambda, 0) \to 0$, but $\mathbf{U}(x, \lambda, t)$ does not tend to zero for $t > 0$. This indicates the absence of a continuous dependence on the initial data. Therefore, with violation of the hyperbolicity conditions, one might expect loss of flow stability in the longwave approximation.

3. Steady Solutions. For consideration of steady solutions, we introduce the average Mach number

$$\mathrm{M} = \Big(\int_0^1 \frac{H\,d\lambda}{\rho^2 c^2}\Big)^{1/2} \Big(\int_0^1 \frac{H\,d\lambda}{\rho^2 u^2}\Big)^{-1/2}. \tag{3.1}$$

If the flow parameters do not depend on λ (homogeneous flow), it is obvious that $\mathrm{M} = |u|c^{-1}$. If the flow is inhomogeneous and $|u(x_0, \lambda)| > c(x_0, \lambda)$ for $x = x_0$, it is not difficult to see that $\mathrm{M} > 1$. Similarly, $\mathrm{M} < 1$ if $|u(x_0, \lambda)| < c(x_0, \lambda)$. Hence, the value $\mathrm{M} = 1$ can be reached at sections $x = \text{const}$ at which $|u| - c$ changes sign. The equality $\mathrm{M}(x_0) = 1$ is equivalent to satisfaction of the characteristic equation for $k = 0$. This means that at the point x_0 one of the perturbation propagation velocities vanishes.

Since the choice of the Lagrangian variable λ is ambiguous, in the steady case it is convenient to set $\lambda = \psi$, where ψ is the stream function ($\psi_y = \rho u$ and $\psi_x = -\rho v$). The difference from the previous choice is insignificant: at the sections $x = \text{const}$, $\lambda = \psi$ changes from 0 to Q, where Q is a constant gas flow rate in the jet. In accordance with this, the limits of integration in (3.1) and similar formulas are changed. In the steady case, Eqs. (1.8) integrate to:

$$S = S(\lambda), \quad 2^{-1}u^2 + i(p, S(\lambda)) = I(\lambda), \quad uH = 1. \tag{3.2}$$

Here S and I are arbitrary functions and $i(p, S)$ is the specific gas enthalpy. Equations (3.2) are solvable for u and H:

$$u = \pm\sqrt{2(I(\lambda) - i(p, S(\lambda)))}, \quad H = \pm(2(I(\lambda) - i(p, S(\lambda))))^{-1/2}. \tag{3.3}$$

Let the entrance velocity u be positive at $x = 0$. Then, the plus sign is fixed in (3.3). Substituting (3.3) into (1.6) yields an equation that defines the pressure $p(x)$:

$$K(p) = \int_0^Q \frac{d\lambda}{\rho(p, S(\lambda))\sqrt{2(I(\lambda) - i(p, S(\lambda)))}} = A(x). \tag{3.4}$$

The derivatives of the function $K(p)$ are of the form

$$K'(p) = (1 - \mathrm{M}^2) \int_0^Q H\rho^{-2}u^{-2}\,d\lambda, \tag{3.5}$$

$$K''(p) = \int_0^Q \frac{\tau_{pp}\,d\lambda}{(2(I - i))^{1/2}} + 3\int_0^Q \frac{\tau\tau_p\,d\lambda}{(2(I - i))^{3/2}} + 3\int_0^Q \frac{\tau^3\,d\lambda}{(2(I - i))^{5/2}} \quad \Big(\tau = \frac{1}{\rho}\Big).$$

It follows from the above formulas that the qualitative behavior of inhomogeneous flows is similar to the behavior of homogeneous flows. For a supersonic flow (M > 1) in an expanding channel $[A'(x) > 0]$, the pressure decreases and the velocity u increases along each streamline. For a subsonic flow (M < 1), p and u behave similarly in a converging channel $[A'(x) < 0]$. We assume that the equations of state of the gas satisfy the monotonicity and convexity conditions: $\tau_p(p,S) < 0$ and $\tau_{pp}(p,S) > 0$. In the case of a homogeneous flow (the solution u, H, and S does not depend on λ), these conditions give the inequality $K''(p_c) > 0$ at the point $p = p_c$ determined by the condition $K'(p_c) = 0$. From this property it follows that $K(p)$ reaches a minimum for $p = p_c$ and changes monotonically for $p \neq p_c$. In the case of inhomogeneous flows, the nonnegativeness condition for the last two terms in the second relation of (3.5) and the condition $K'(p_c) = 0$ can be represented as

$$\int_0^Q (\tau_p + \tau^2 f^2)\tau f^3\, d\lambda \geqslant 0; \tag{3.6a}$$

$$\int_0^1 (\tau_p + \tau^2 f^2) f\, d\lambda = 0, \tag{3.6b}$$

where $f = (2(I-i))^{-1/2}$. In the general case, inequality (3.6a) is not a consequence of equality (3.6b), but for the equation of state

$$\tau = b(S)\varphi(p) \tag{3.7}$$

($b > 0$, $\varphi' < 0$, and $\varphi'' > 0$) inequality (3.6a) follows from (3.6b). In this case, inequality (3.6a) reduces to the Cauchy inequality

$$\Big(\int_0^Q b^2 f^3\, d\lambda\Big)^2 \leqslant \int_0^Q b^3 f^5\, d\lambda \int_0^Q bf\, d\lambda.$$

We distinguish a class of gas equations of state for which the function $K(p)$ is convex for all values of p. These equations of state are characterized by the inequality

$$4\tau\tau_{pp} - 3\tau_p^2 > 0. \tag{3.8}$$

For these functions $\tau(p,S)$ the integrand in the second formula of (3.5) is positive due to the inequality $ab > -a^2 - 4^{-1}b^2$, where $a = \tau^{3/2}(2(I-i))^{-5/4}$ and $b = \tau_p\tau^{-1/2}(2(I-i))^{-1/4}$. Hereinafter we assume that the equations of state of the gas are of the form (3.7) or satisfy the condition (3.8). Then, there is a single value $p = p_c$ for which the function $K(p)$ reaches a minimum and $K(p)$ varies monotonically for $p \neq p_c$.

We consider a steady gas flow through a given channel. At the entrance at $x = 0$, we assume $u = u_0(y) > 0$, $p = p_0 = \text{const} > 0$ (p_0 is the average pressure at the entrance), and $\rho = \rho_0(y) > 0$. At $x = 0$, these conditions determine

$$\psi_0(y) = \int_0^y \rho_0(y')u_0(y')\, dy', \quad Q = \int_0^{A(0)} \rho_0(y')u_0(y')\, dy'. \tag{3.9}$$

The function $S_0(y)$ is obtained from the equation $\rho(p_0, S_0(y)) = \rho_0(y)$ [it is assumed that $\rho_S(p,S) > 0$]. In accordance with (3.1), the entrance flow is subsonic (M < 1) if

$$\int_0^{A(0)} \frac{dy}{\rho_0(\eta)u_0^2(\eta)} > \int_0^{A(0)} \frac{dy}{\rho_0(\eta)c^2(p_0,\rho(\eta))},$$

or supersonic when the inverse inequality holds. Solving the first equation of (3.9) for $y = y_0(\psi)$, we find $I(\psi) = I(\lambda) = 2^{-1}u_0^2(y_0(\psi)) + i(p_0, S_0(y_0(\psi)))$. This allows us to define the function $K(p)$ by formula (3.4)

From the equation

$$K'(p) = \int_0^Q \frac{-c^{-2}(p, S(\psi)) + (2(I(\psi) - i(p, S(\psi))))^{-1}}{\rho^2(p, S(\psi))(2(I(\psi) - i(p, S(\psi))))^{1/2}}\, d\psi = 0, \tag{3.10}$$

where $S(\psi) = S_0(y_0(\psi))$, we find the critical pressure $p = p_c$. In this case $p_c < p_0$ if $\mathrm{M} < 1$ at the entrance and $p_c > p_0$ if $\mathrm{M} > 1$ at the entrance. Indeed, the condition $\mathrm{M} < 1$ is equivalent to the inequality $K'(p_0) > 0$, but as $p \to 0$, the integrand in (3.10) becomes negative $[c(p, S(\psi)) \to 0$ for $p \to 0]$. Therefore, $K'(p)$ changes sign on the interval $(0, p_0)$ and $0 < p_c < p_0$. We note that relations (3.3) are defined for $p > p^0 = \max\limits_{\psi} P(\psi)$, where $P(\psi)$ is a root of the equation $I(\psi) = i(P(\psi), S(\psi))$. If at the channel entrance $\mathrm{M} > 1$, we have $K'(p_0) < 0$, but $K'(p) \to +\infty$ for $p \to p^0$. Therefore, $K(p)$ changes sign on the interval (p_0, p^0) and $p_0 < p_c < p^0$. The inequality $A(x) \geqslant K(p_c) = \min\limits_{p} K(p)$ must be satisfied for the existence of continuous steady flow; otherwise, Eq. (3.4) has no solutions at certain x.

From the properties of the function $K(p)$ it follows that in supersonic flow regions, $p < p_c$ and in subsonic regions, $p > p_c$. Therefore, the radicand in (3.3) $(p = p^0)$ can vanish only in a subsonic region of an expanding portion of the channel $[A(x)$ takes the value $K(p^0)]$. If the function $A(x)$ satisfies the inequality $K(p_c) \leqslant A(x)$ everywhere, from Eq. (3.4) one obtains the pressure distribution $p = p(x)$ and then $u(\lambda, x)$ and $H(\lambda, x)$ using formulas (3.3).

Note that at points x_i at which the equality $p(x_i) = p_c$ holds, branching of the solution is possible: if $A(x) > K(p_c)$ for $x > x_i$, the solution $p = p(x)$ of Eq. (3.4) can be continued to the region $x > x_i$ either by a subsonic branch $(p > p_c)$ or by a supersonic branch $(p < p_c)$. Thus, we can also construct transonic flows, including flow through a nozzle with a throat of width $K(p_c)$ with subsonic flow in the convergent portion and supersonic flow in the divergent portion. If for a chosen steady flow in a divergent subsonic portion of the channel, the equality $A(x) = K(p^0)$ is reached with increase in x, solutions of the form (3.3) cease to exist. In a supersonic portion of the channel, attainment of the equality $A(x) = K(p^0)$ does not prevent further continuation of solution (3.3), because in this portion the pressure decreases with expansion of the channel $(p < p_c < p^0)$.

4. Return Flows. The stagnation point of the flow can appear, in particular, in a completely subsonic gas flow through an asymmetric expanding channel. Let the function $A(x)$, which defines the shape of the upper wall of the channel, satisfy the inequality $A(x) > K(p_c)$ for the subsonic flow considered. In addition, let the inequality $A(x) < K(p^0)$ be satisfied for $x < x_1$ and the equality $A(x_1) = K(p^0)$ hold at the point $x = x_1$. For definiteness, we assume that $u > 0$ for $x < x_1$ and, at $x = x_1$, the function u becomes zero for the first time on the upper wall of the channel. Since for subsonic flow, $K'(p) > 0$, solutions of the form (3.3) cease to exist for $x > x_1$ if the channel continues to expand $[A(x) > A(x_1)]$. But for $x > x_1$, we can construct a steady solution of another structure. For this solution, a dividing streamline A_1B_1 issues from the point A_1, where $u = 0$ (Fig. 1). Particles that enter the left end of the channel move below this streamline, and turning of the streamlines — the trajectories of the particles entering the right end of the channel — occurs above the line. This solution describes flow with separation of the main flow from the channel wall. Let the streamline A_1B_1 be given by the equation $y = \eta(x)$ $(x > x_1)$.

We examine the cross section $x = \text{const}$, $x > x_1$. Since, in the main-flow and return-flow regions, relations of the form (3.3) are satisfied for the appropriate choice of the radical sign, for the main-flow region we have

$$\eta(x) = \overline{\eta}(p(x)) = \int_0^Q \frac{d\lambda}{\rho(p, S(\lambda))\sqrt{2(I(\lambda) - i(p, S(\lambda)))}}.$$

In the region between A_1B_1 and the upper wall of the channel, the flow is described by formulas (3.3), where one must replace $I(\lambda)$ by $I_1(\lambda)$, $S(\lambda)$ by $S_1(\lambda)$, $i(p, S)$ by $i_1(p, S_1)$, and $\rho(p, S)$ by $\rho_1(p, S_1)$ (the gas entering the channel from the right generally has a different equation of state and different flow "constants"). The stagnation points of the flow have coordinates $(x, \lambda(p(x))$, where $\lambda(p)$ is a solution of the

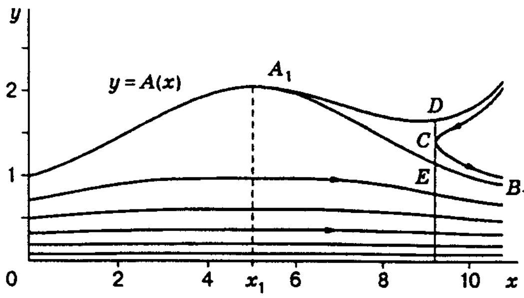

Fig. 1

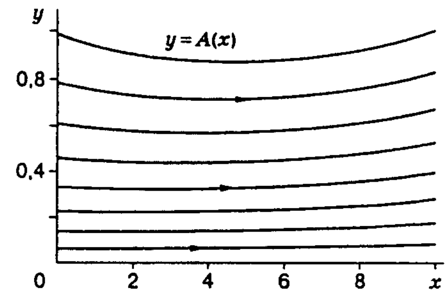

Fig. 2

equation $I_1(\lambda(p)) - i_1(p, S_1(\lambda)) = 0$.

On the segment EC, where λ varies from $\lambda(p^0) = Q$ to $\lambda(p)$, the plus sign is chosen in (3.3). On the segment CD, where λ varies from $\lambda(p)$ to Q, the minus sign is chosen. The thickness of the layer occupied by the return flow $\xi = \xi(x)$ is written in the form

$$\xi(x) = \bar{\xi}(p(x)) = \int\limits_{\varphi}^{\lambda(p)} \frac{d\lambda}{\rho_1\sqrt{2(I_1 - i_1)}} - \int\limits_{\lambda(p)}^{Q} \frac{d\lambda}{\rho_1\sqrt{2(I_1 - i_1)}}.$$

The equality $\bar{\eta}(p) + \bar{\xi}(p) = A(x)$ defines the pressure distribution along the channel at $x > x_1$ if $I(\lambda)$, $I_1(\lambda)$, $S(\lambda)$, and $S_1(\lambda)$ are known. If the function $p = p(x)$ is known, the channel flow for $x > x_1$ is described by relations of the form (3.3), as noted above.

For consideration of particular examples, for simplicity we set $S_1(\lambda) = S_1 = \text{const}$. Assuming that the function $I_1(\lambda)$ is monotonic, in the integral representing $\bar{\xi}(p)$ we convert to the integration variable $\mu = I_1(\lambda)$. We introduce the function $w(\mu) = (I_1'(\lambda))^{-1}$. Then the equation $\bar{\xi}(p) + \bar{\eta}(p) = A(x)$ is rewritten as

$$2\int\limits_{I_1(Q)}^{i_1(p,S_1)} \frac{w(\mu)\,d\mu}{\rho_1(p,S_1)\sqrt{2(\mu - i_1(p,S_1))}} = A(x) - \bar{\eta}(p).$$

The return-flow region is completely specified by defining the equations of state, the function $w(\mu)$, and the constant S_1. We examine the simple case $w(\mu) = -N = \text{const}$. In this case, the previous equation is simplified:

$$\bar{\xi}(p) = 2N(\rho(p,S_1))^{-1}\sqrt{2(I_1(Q) - i_1(p,S_1))} = A(x) - \bar{\eta}(p). \tag{4.1}$$

Here $I_1(Q) = i_1(p^0, S_1)$ by virtue of the fact that u vanishes for $p = p^0$. For values of p close to p^0, we have $\bar{\xi}'(p) + \bar{\eta}'(p) < 0$. Therefore, the function $p(x)$, which decreases monotonically in a vicinity of the point $x = x_1$, is determined from Eq. (4.1) for $A'(x) > 0$. In the indicated vicinity, the thickness of the layer occupied by the return flow $\bar{\xi}(p(x))$ increases and the thickness of the layer occupied by the main flow decreases. In an expanding channel, the pressure continues to decrease as long as $\bar{\xi}'(p) + \bar{\eta}'(p) < 0$. Note that at the point x_1 the pressure reaches a maximum value $p = p^0$.

We consider results of numerical simulation of inhomogeneous polytropic-gas flows through a nozzle based on the approximate model.

We assume that in relations that define conversion to dimensionless variables, $U = C_0$, where C_0 is the entrance velocity of sound at $x = 0$ and $y = 0$, H_0 is the initial width of the channel entrance, and R_0 is the gas density at $x = 0$ and $y = 0$. Then, the dimensionless flow parameters for $x = y = 0$ satisfy the relations $p = \gamma^{-1}$, $\rho = 1$, and $A(0) = 1$, where γ is the polytropic index. The equation of state of the gas is written as $p = \gamma^{-1}a(S)\rho^\gamma$ [$a(S) = 1$ for $x = y = 0$]. The channel entrance flow is characterized by a constant pressure, a constant horizontal velocity component, and a temperature profile linear in y:

$$u = u_0 = \text{const}, \quad \rho = (1 + \beta u_0 y)^{-1}, \quad a(S) = (1 + \beta u_0 y)^\gamma, \quad p = \gamma^{-1}$$

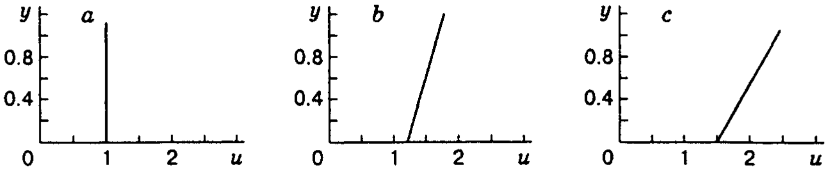

Fig. 3

($u_0 > 0$ and $\beta > 0$). Then, for $x = 0$, we have

$$\psi = \psi(y) = \beta^{-1} \ln(1 + \beta u_0 y), \quad a(S(\psi)) = e^{\beta\gamma\psi}, \quad Q = \psi(1).$$

These relations define $I(\psi)$ and $i(p, S(\psi))$:

$$I = 2^{-1}u_0^2 + (\gamma - 1)^{-1}e^{\beta\psi}, \quad i = (\gamma p)^{(\gamma-1)/\gamma}e^{\beta\psi}/(\gamma - 1).$$

In accordance with the previous formulas, $u(p, \psi)$ and $y(p, \psi)$ for $x > 0$ are given by the equalities

$$u(p, \psi) = \sqrt{u_0^2 + 2(\gamma - 1)^{-1}e^{\beta\psi}(1 - (\gamma p)^{(\gamma-1)/\gamma})},$$

$$y(p, \psi) = 2\beta^{-1}(\gamma p)^{-1/\gamma}(e^{\beta\psi} - 1)(u(p, \psi) + u(p, 0))^{-1},$$

and the pressure distribution is found from the equation $y(p, Q) = A(x)$. The entrance flow is subsonic (M < 1) if $u_0 < 4^{-1}\left(\beta + \sqrt{\beta^2 + 16}\right)$, otherwise it is supersonic (M > 1).

Figure 2 shows streamlines, and Fig. 3a–c shows, respectively, the velocity profiles u at the entrance, throat (M $= 1$), and exit of the nozzle for transonic steady flow with a linear pressure distribution $p = \gamma^{-1}(1 - bx)$ along the channel [$\gamma = 1.4$, $u_0 = 1$, $\beta = 3$, and $b = (16)^{-1}$]. The temperature inhomogeneity of the subsonic entrance flow leads to supersonic shear flow at the nozzle exit.

Figure 1 shows the streamline pattern for subsonic flow with the same channel entrance flow parameters for the following pressure distribution along the channel:

$$p = (5/7)((21/20)^{7/2} - ((21/20)^{7/2} - 1)(x/5 - 1)^4).$$

In an expanding channel at $x < 5$, the subsonic flow decelerates, the horizontal velocity profile becomes inhomogeneous along the vertical, and, at $x = 5$, a stagnation point of the flow occurs at the upper wall. For $x > 5$, the steady flow continues as flow with separation of the main flow from the wall and formation of a return-flow region. It is assumed that in the return-flow region, the gas is described by the same equation of state and its entropy is constant; its value coincides with $S(Q)$ (the value of S in the main flow on the upper streamline). The return-region flow is characterized by the additional relation $w(\mu) = -N = -0.1$, which corresponds to the linear velocity profile $u(y)$ on the inflow portion at the right end of the channel $x = 11$. Expansion of the return-flow region leads to a decrease in the thickness of the region occupied by the main flow and an increase in the velocity of the main flow.

In summary, a longwave approximation model is constructed that describes unsteady inhomogeneous gas flows in a long channel of varying area. It is shown that the system of equations of motion is hyperbolic under certain conditions. The general properties of steady gas flows with equations of state that satisfy the monotonicity and convexity conditions and the additional condition (3.7) or (3.8) are studied.

In considering more general equations of state that satisfy the monotonicity and convexity conditions, one might expect changes in the qualitative properties of steady flows [the presence of several maxima and minima of the function $K(p)$].

Remark. In the case of an axisymmetric channel flow [$0 \leqslant r \leqslant B(x)$], the equations of the same approximation in Eulerian variables are of the form

$$u_t + uu_x + wu_r + \rho^{-1}p_x = 0, \; p_r = 0, \; \rho_t + (u\rho)_x + r^{-1}(\rho r w)_r = 0, \; S_t + uS_x + wS_r = 0. \qquad (4.2)$$

Here x is the dimensionless coordinate along the channel axis, r is the dimensionless radial coordinate in a cylindrical coordinate system, and u and w are the corresponding velocity components [the dimensionless variables are introduced in the same manner as in (1.1a)]. It is easy to see that after the change of variables $y = 2^{-1}r^2$, $v = rw$, and $A(x) = 2^{-1}B^2(x)$, Eqs. (4.2) become (1.1) at $\varepsilon = 0$, and the boundary condition on the channel wall becomes (1.3). Therefore, all the results are also valid for an axisymmetric flow, and exact solutions are obtained from solutions of the plane problem by substitution of variables.

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